Math 472 **Statistics** **Spring 2008**

Homework Set 1 , given Fri 2/1, due Friday 2/8

Problem 1. Suppose that $X_1, \ldots, X_n$ is a set of r.v.'s all having the same finite range of possible values: $\mathcal{X} = \{x_1, \ldots, x_m\}$, i.e we have $X_i \in \mathcal{X}$, $i = 1, \ldots, n$. Define independence of $X_1, \ldots, X_n$ by the property: for all n-tuples $(x_1, \ldots, x_n) \in \mathcal{X}^n$ we have

$$\Pr(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} \Pr(X_i = x_i).$$

Show that independence of $X_1, \ldots, X_n$ implies independence of any subset $X_{i_1}, \ldots, X_{i_k}$ where $\{i_1, \ldots, i_k\}$ is an arbitrary subset of size k of the indices $\{1, \ldots, n\}$, and $2 \leq k < n$.

Problem 2. Let $X_1, X_2 \ldots$ be a sequence of r.v.'s and μ be a real number. Let Y be a r.v. taking value μ with probability one ($\Pr(Y = \mu) = 1$). The law, or probability distribution, of Y is called the *degenerate law concentrated at* μ; it has distribution function

$$G_\mu(x) = \Pr(Y \leq x) = \begin{cases} 0, & x < \mu \\ 1, & x \geq \mu. \end{cases}$$

Show that as $n \to \infty$

$$X_n \rightsquigarrow Y \text{ if and only if } X_n \longrightarrow_p \mu,$$

i.e. convergence in distribution to the degenerate law means convergence in probability to μ. For these convergence notions, see manuscript (handout 1/23, definition of $\rightsquigarrow$, and the well known convergence in probability: $X_n \longrightarrow_p \mu$ if $\Pr(|X_n - \mu| \geq \varepsilon) \to 0$ for every $\varepsilon > 0$).

Problem 3. Suppose X, Y are independent r.v.'s with Poisson distributions, specifically $\mathcal{L}(X) = \mathrm{Po}(t)$, $\mathcal{L}(Y) = \mathrm{Po}(u)$ where $t, u > 0$. Show that

$$\mathcal{L}(X + Y) = \mathrm{Po}(u + t).$$

Problem 4. The proof of the CLT on p. 228 [D][1] is given under the assumption that the i.i.d. r.v.'s involved $X_1, X_2, \ldots$ have an exponential moment, more precisely that there exists $t_0 > 0$ such that $E \exp(tX) < \infty$ for $t \in (-t_0, t_0)$. Show that this condition is fulfilled if the X_i have a $\mathrm{Po}(\lambda)$ law. (Comment: in lecture we argued that $\mathrm{Po}(n)$ has a normal approximation as $n \to \infty$, based on the representation as a sum of i.i.d. Poisson $\mathrm{Po}(1)$ variables X_i)

Problem 5. Suppose the probability of having blue eyes is 0.15 for any given person in the U.S.. The town of Springfield, USA has 800 people. Suppose the residents of Springfield are all unrelated and have eye colors that are independent of each other.

- **a):** Find the expected number of people with blue eyes in Springfield, USA.
- **b):** Find the variance and standard deviation of the number of blue-eyed people in Springfield, USA.

[1][D]: Durrett, R., *The Essentials of Probability,* Duxbury Press, 1994.

c): Use the Normal approximation to the Binomial to calculate the probability that there are between 110 and 125 blue-eyed residents of Springfield, USA. Be sure to verify the success/ failure condition for validity of the normal approximation.